Stabilnost rešenja linearnih sistema diferencijalnih j-na

$X' = A(t)X + B(t)$ nehomogeni linearni sis. DJ (1)

$G = I \times \mathbb{R}^n$

Teorema: Sva rešenja sistema linearnih DJ-a su istovremeno ili stabilna ili nestabilna.

Posljedica: Bilo koje rešenje nehomogenog sistema linearnih DJ-a (1) je stabilno po Ljapunovu $\Leftrightarrow$ je stabilno po Ljapunovu trivijalno rešenje sistema $X' = A(t) \cdot X$.

– važi i za asimptotsku stabilnost

Sistemi sa konst. koef.

$X' = AX$ (2)

A – matrica sa const. koef.

$\lambda_j = \alpha_j + i\beta_j$ – svojstvene vrijednosti matrice A

Teorema: Ako postoji $\alpha > 0$ t.d. $\alpha_j < -\alpha$ $\forall j$, tada svako rj. sistema (2) $x = \varphi(t)$ kqe zad $\varphi(t_0) = \varphi_0$ je asimpt stabilno.

Teorema: Ako $\exists \alpha > 0$ t.d. $\exists j$ $\alpha_j > \alpha$, onda je nestabilno.

Teorema: Ako je $\alpha_k = 0$ za proste, višestrukosti 1, λ_k i $\alpha_k < 0$ za ostale $\Big\}$ $\Rightarrow$ $x = 0$ stabilno.

Teorema: $p(\lambda) = \det(A - \lambda E) = 0$ je stabilan $\Leftrightarrow$ $x = 0$ je asimptotski stabilan.

Lema Raus-Hurvica:

$p(\lambda) = \lambda^n + a_1 \cdot \lambda^{n-1} + \dots + a_n$ je stabilan $\Leftrightarrow$ su svi glavni minori matrice pozitivni ($a_k = 0$ za $k > n$)

$$\begin{array}{ccccc} \text{nep} & \text{par} & \text{nep} & & \\ a_1 & 1 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \vdots \\ a_5 & a_4 & a_3 & a_2 \dots & \vdots \\ & & & & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & & a_n \end{array}$$

$a_k = 0$, $k > n$

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Lema (Lema Šipar): $p(\lambda)$ je stab. $\Leftrightarrow a_i > 0$, $\Delta_{n-1} > 0$, $\Delta_{n-3} > 0, \ldots$

1. $x_1' = -x_1 + \alpha x_2 + \beta x_3$

$x_2' = -\alpha x_1 - x_2 + \alpha x_3$

$x_3' = -\beta x_1 - \alpha x_2 - x_3$

U zavisnosti od α i β, odrediti kad je stabilno rešenje.

R:

$$A - \lambda E = \begin{vmatrix} -1-\lambda & \alpha & \beta \\ -\alpha & -1-\lambda & \alpha \\ -\beta & -\alpha & -1-\lambda \end{vmatrix} = 0$$

$$= -\lambda^3 - 3\lambda^2 + \lambda(-3-\beta^2-2\alpha^2) + (-1-2\alpha^2-\beta^2)$$

Posmatramo polinom:

$$\lambda^3 + \underbrace{3}_{a_1}\lambda^2 + \lambda\underbrace{(3+\beta^2+2\alpha^2)}_{a_2} + \underbrace{(1+2\alpha^2+\beta^2)}_{a_3}$$

$$\begin{vmatrix} 3 & 1 & 0 \\ 1+2\alpha^2+\beta^2 & 3+\beta+2\alpha^2 & 3 \\ 0 & 0 & 1+2\alpha^2+\beta^2 \end{vmatrix}$$

$\Delta_1 = 3 > 0$

$\Delta_2 = 9 + 3\beta + 6\alpha^2 - 1 - 2\alpha^2 - \beta^2 = 8 + 4\alpha^2 + 2\beta^2 > 0$

$\Delta_3 = \left(1 + 2\alpha^2 + \beta^2\right)\Delta_2 > 0$

Polinom je stabilan $\forall \alpha$ i $\forall \beta$.

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$$\begin{cases} x_1' = a_{11} x_1 + a_{12} x_2 \\ x_2' = a_{21} x_1 + a_{22} x_2 \end{cases}$$

<— Na osnovu teorema, zaključujemo:

sedlo = nestabilno

centar $\pm i\beta$ = stabilan

stabilni čvor $\lambda_1 < \lambda_2 < 0$

stabilan fokus $\alpha \pm i\beta$, $\alpha < 0$

} asimp. stabilni

(2.) $x' = y$

$y' = -x$

$z' = -z$

R:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-\lambda(-1-\lambda)) - 1(1+\lambda) = -\lambda^2(1+\lambda) - (1+\lambda) = -(\lambda^2+1)(1+\lambda) = 0$$

$\underline{\underline{\pm i}}$ $\underline{\underline{-1}}$

(T) $\Rightarrow$ stabilnost

(3.) $x_1' = x_3$

$x_2' = -3x_1$

$x_3' = a x_1 + 2x_2 - x_3$

U zavisnosti od a kad je asimp. stab.

R:

$$p(\lambda) = -\lambda^3 - \lambda^2 + a\lambda - 6 \longrightarrow \lambda^3 + \lambda^2 - a\lambda + 6$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 6 & -a & 1 \\ 0 & 0 & 6 \end{pmatrix}$$

$\Delta_1 > 0$

$\Delta_2 = -a - 6 > 0 \iff \boxed{a < -6}$

↳ asimptotski stabilno

$\Delta_3 = 6 \cdot \Delta_2$

④ Pokazati da ako je svako reš. lin. homog. sis. ograničeno za $t \geq 0$, da je tada trivijalno rešenje stabilno po Lj.

D:

Čim imamo reš. ⇒ imamo $\Phi(t)$

$\Phi(0) = \mathbb{I}$

tražimo da ovo bude jedinična matrica

$$\begin{cases} x(t) = \Phi(t) \cdot x_0 \\ x(0) = x_0 \end{cases}$$ Košijevo rešenje koje zadovoljava početni uslov

$\boxed{x = \Phi(t) \cdot c}$ – opšti oblik rješenja

$\exists M \quad \|\Phi(t)\|_2 \leq M$ (ograničeno rešenje)

$\|x(t)\|_2 = \|\Phi(t) \cdot x_0\|_2 \leq \|\Phi(t)\|_2 \cdot \|x_0\|_2 \leq M \cdot \|x_0\| < \varepsilon$

Uzmemo $\delta = \dfrac{\varepsilon}{M}$ ...

⑤ $x_1' = x_1 - x_2 - x_3$

$x_2' = x_1 + x_2 - 3x_3$

$x_3' = x_1 - 5x_2 - 3x_3$

R:

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & -3 \\ 1 & -5 & -3-\lambda \end{vmatrix} = 0$$

$((1-\lambda)((\lambda-1)(\lambda+3) - 15) + 1(\cancel{-3} - \lambda + \cancel{3}) - 1(-5 - 1 + \lambda) =$

$(+1-\lambda)(\lambda^2 + 2\lambda - 18) - 2\lambda + 6 =$

$\lambda^2 + \cancel{2\lambda} - 18 - \lambda^3 - 2\lambda^2 + 18\lambda - \cancel{2\lambda} + 6 = 0$

$-\lambda^3 - \lambda^2 + 18\lambda - 12 = 0$

$\lambda^3 + \lambda^2 - 18\lambda + 12 = 0$

$a_1 \quad a_2 \quad a_3$

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$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 12 & -18 & 1 \\ 0 & 0 & 12 \end{vmatrix}$$

$\Delta_1 > 0$

$\Delta_2 < 0$

↳ znamo da nije asimp. stabilan, a da li je stabilan?

$p(3) = 27 + 9 - 54 + 12 < 0$

$p(4) = 64 + 16 - 72 + 12 > 0$

Koristeći To međuvr. $\Longrightarrow$ $\exists c \in (3,4)$ t.d. $p(c) = 0$, $c > 0$

$x = 0$

$\Rightarrow$ ~~polinom~~ nije stabilan

⑥ $x_1' = ax_1 - x_2$

$x_2' = ax_2 - x_3$

$x_3' = ax_3 - x_1$

⟵ u zavisnosti od a ispitati stabilnost rešenja sistema DJ

$$\det(A - \lambda E) = \begin{vmatrix} a-\lambda & -1 & 0 \\ 0 & a-\lambda & -1 \\ -1 & 0 & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a-\lambda)^3 + 1 \cdot (-1) = 0$

$(\lambda - a)^3 + 1^3 = 0$

$\lambda - a = \sqrt[3]{-1} = 1 \cdot e^{\frac{\pi + 2k\pi}{3} \cdot i}$, $k \in \{0,1,2\}$

$k = 0 \Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$k = 1 \Rightarrow -1$

$k = 2 \Rightarrow \frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\lambda_1 = a - 1$ ⟵ nule

$\lambda_2 = (a + \frac{1}{2}) + i\frac{\sqrt{3}}{2}$

$\lambda_3 = (a + \frac{1}{2}) - i\frac{\sqrt{3}}{2}$

$(\lambda - a + 1)((\lambda - a)^2 - (\lambda - a) + 1) = 0$

$\lambda_1 = a - 1$

$\lambda^2 + \lambda(-2a - 1) + a^2 + a + 1 = 0.$

$\lambda_{2,3}$

Da bi bio asimptotski stabilan

$a - 1 < 0$

$a + \frac{1}{2} < 0$

$\Downarrow$

$\boxed{a < -\frac{1}{2}}$ tada je asimp. stabilan

$a > -\frac{1}{2}$ $\Rightarrow$ nestabilno

$a = -\frac{1}{2}$

$$\left.\begin{aligned} \lambda_1 &= -\frac{3}{2} \\ \lambda_{2,3} &= \pm i \frac{\sqrt{3}}{2} \end{aligned}\right\}$$ stabilno (ali nije asimp. stabilno)

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Stabilnost položaja ravnoteže

sistema j-na na osnovu linearizacije

$X = 0$

$X' = F(X)$

$F \in C^2(G(0))$ (okog položaja ravnoteže)

$F(X) = \underbrace{F(0)}_{=0} + F'(0) \cdot X + g(X)$ pri čemu $\|g(X)\| \le c \cdot \|X\|^2$

Teorema:

$F \in C^2(G(0))$

$X = 0$ položaj ravnoteže $\qquad X' = F'(0) \cdot X$

Ako je $\det(F'(0) - \lambda E)$ stabilan tada je $X = 0$ asimptotski

stabilan za sistem $X' = F(X)$.

Realni dio bar jedne nule $p(\lambda) = \det(F'(0) - \lambda E) > 0 \Rightarrow X = 0$ nestabilan linearni dio

1. $x_1' = \underbrace{-x_1 + x_2}_{} + \underbrace{2x_1x_2}_{R_1}$

$x_2' = \underbrace{2x_1 - 3x_2}_{} + \underbrace{5x_1^4 + x_2^5}_{R_2}$

$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ gledamo ovaj položaj ravnoteže $\qquad X = (x_1, x_2)$

$|R_1| = |2x_1x_2| \leq x_1^2 + x_2^2 = \|X\|^2 \quad , c = 1$

$|R_2| = |5x_1^4 + x_2^5| \leq 5|x_1|^4 + |x_2|^5 \leq 5|x_1|^4 + 5|x_2|^5$

$|x_2|^5 < |x_2|^4$ (možemo ovo da napravimo) $\Leftrightarrow |x_2| \leq 1$

$\leq 5|x_1|^4 + 5|x_2|^4 \leq 5x_1^4 + 5x_2^4 + 10x_1^2x_2^2 = 5(x_1^2 + x_2^2)^2 = 5\|X\|^4$

$\leq \ldots \leq 5\|X\|^2 \quad , c = 5 \qquad (\|X\| < 1)$

$\Rightarrow$ Možemo preći na linearni sistem

$x_1' = -x_1 + x_2$

$x_2' = 2x_1 - 3x_2$

$A = \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}$

$\det(A - \lambda E) = 0$

$(-1-\lambda)(-3-\lambda) - 2 = 0$

$\lambda^2 + 4\lambda + 1 = 0$

$\lambda_{1/2} = \dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3} < 0$ (stabilan je jer su obije tačke negativne)

$\Rightarrow$ Teorema $\Rightarrow$ asimp. stabilan $x = 0$

$X' = A(t)X + B(t)$

$X' = A(t)X$

$\underline{X = 0}$ $\quad X(t_0) = 0.$

$\varphi = \varphi(t), \quad \psi(t_0) = \psi_0.$

$\psi(t) = X_H(t) + \varphi(t)$

$\psi(t_0) = \psi_0 = X_H(t_0) + \varphi_0.$

$\| \varphi(t_0) - \psi(t_0) \| < \delta \quad \Leftrightarrow \quad \| X_H(t_0) \| < \delta$

$\Rightarrow \| \varphi(t) - \psi(t) \| < \varepsilon \quad \Leftrightarrow \quad \| X_H(t) \| < \varepsilon$

$\phi(t) = [\phi_1(t), \dots, \phi_n(t)]$

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